1 INTRODUCTION

Limit equilibrium methods of slope stability analysis are easily operated and widely used in practical slope engineering [1]. In 1916, Petterson and Hultin proposed circle sliding surfaces for stability analysis of undrained soil slopes [2], which is later called the Swedish circle method and marks the beginning of the application of limit equilibrium method for slope stability. In 1926, Fellenius introduced both cohesive and frictional strength of soil into slope stability analysis using the circular sliding surface. Thus, the Fellenius method...
is regarded as one of the significant limit equilibrium methods with fully considering soil strength for slope stability analysis [3]. Later, Lorimer once used Euler spiral or clothoid instead of circular slip surface to explain failure mode of soil slope by referring to experiments and investigations. In these methods, potential slide mass is divided vertically into finite slices with small width. And interslice forces on the two sides of a slice are assumed to be a pair of balance forces. Bishop (1955) [4] improved further the assumption of interslice force with neglecting the difference between their vertical components and proposed the simplified Bishop method for the soil slope with circular slip surface. Janbu (1954) [5] also neglected the tangent component of interslice force but provided the simplified Janbu method used for the soil slope with arbitrary slip surface. In order to reduce the disadvantage of the previous methods without considering the tangent component of interslice force, Lowe and Karafiath (1960) [6] assumed that the dip angle of the interslice force is equal to the average of the dip angle of the top and bottom of the slice. The US Army Corps of Engineers (1967) [7] recommended that the dip of interslice force is equal to the average of the dip angles of all slices (Corps of Engineers #1) or the dip of the top of the slice (Corps of Engineers #2). However, for the Imbalanced Thrust Force Method (ITFM) [8], the dip angle of the interslice force is assumed to be consistent with the dip of the bottom of the previous slice. Since these methods mentioned above cannot include all equilibrium equations of force and moment of a slice, analysis results of slope stability by them sometimes are not possibly certainly reasonable [9].

Morgenstern and Price (1965) [10] proposed a general slice method which satisfies both force and moment equilibrium by assuming that the ratio of tangential over normal component of interslice force is a specified function. Spencer (1967) [11] presumed that the ratio is an unknown constant. In fact, it is a special case of the Morgenstern-Price method (MPM). Janbu (1973) [12] obtained the solution of slice method by considering all static equilibrium conditions and assumed the location of action point of the interslice force. Fredlund and Krahn (1977) [13] provided an alternative derivation for the Morgenstern-Price method and fully exhibited the relationship between factor of safety and the scaling parameter defined in the MPM. Sarma (1979) [14] proposed a method of non-vertical slicing with considering interfaces between adjacent slices to be in the limit state, which is widely used for stability analysis of rock slope. Chen and Morgenstern (1983) [15] improved the MPM and promoted the function of interslice force further to approximate practical condition. Correia (1988) [16] assumed the shear force on the slice sides to be a function characterizing the shape of the shear force across the slice mass multiplied by a scaling parameter with the unit of force. Zhu (2001) [17] presented a new concise formulation of force and moment equilibrium equations within the framework of the MPM. Although these methods involved completely in both force and moment equilibrium conditions of slices, they all stemmed from assumptions of the direction or magnitude of interslice force.

Therefore, rationality of slice methods of limit equilibrium depends to great extent on whether interslice force is reasonably coped with. Rigorously speaking, although some investigations [9,15] indicate assumptions of interslice force tend to have possibly small effects on the results of the slice method, the minimum value of the factor of safety of slope stability by the slice method in theory has not been completely demonstrated.

From the perspective of rigorous theory, in spite of various assumptions of interslice force adopted in the previous studies, acceptable hypotheses of interslice force based on the practically physical sense are that: (1) The shear forces on the sides of slices should not exceed the related shear resistances; (2) There is no tensile stress between adjacent slices [9]. On the basis of the two elementary constrain conditions, this paper provided a more rigorous derivation for the slice method satisfying all static equilibrium conditions of slices and boundary conditions of the slope without introducing any prescribed relationship between the tangential and normal force on the slice sides. As far as the calculation procedure in mathematics is concerned, the slope stability analysis is transformed rigorously into a non-linear programming problem to ensure the solution of the minimum factor of safety.

2 BASIC PROCEDURE

A typical analysis model of slope stability by slice method is shown in Fig. 1. The key points of slope stability analysis can be regarded essentially as solving the unknown forces on potential sliding surfaces and sides of slices under various factors of safety. Based on the general slice method [10], each slice must satisfy the static equilibrium conditions. Namely, there are two force equilibrium and one moment equilibrium equations for each slice.

Therefore, for the ith slice the three static equilibrium equations can be expressed as (the meaning of all symbols are explained in the Notation)

\[ E_i + N_i \cdot \sin \alpha_i = E_{i+1} + T_i \cdot \cos \alpha_i \]  
(1)

\[ W_i + X_i = X_{i+1} + N_i \cdot \cos \alpha_i + T_i \cdot \sin \alpha_i \]  
(2)
Besides, the bottom of each slice is in the limit state. So according to the Mohr-Coulomb strength criterion and shear strength reduction strategy [18] for the factor of safety, there is

\[ T_i = \left( N_i \cdot \tan \varphi_i + c_i \cdot l_i \right) / F_s \quad \text{(4)} \]

In fact, vertical interfaces subjectively divided between slices of potential slide soil mass are not certainly in the limit state, which is different from the hypothesis assumed by Sarma (1979) [14]. In other words, shear force on the sides of the slice is not over the related shear resistance. Namely,

\[ X_i \leq (E_i \cdot \tan \varphi_i + c_i \cdot d_i) / F_s \quad \text{(5)} \]

Introducing a non-negative coefficient \( K_i \) no more than 1, Eq. (5) can be rewritten as

\[ X_i = K_i \cdot (E_i \cdot \tan \varphi_i + c_i \cdot d_i) / F_s \quad \text{(6)} \]

where \( E_i \) is not less than zero due to the fact that there is no tension between slices.

We can suppose that there is a thin transition layer with a small thickness of \( \delta \) between two adjacent slices (see Fig. 2). And there are average tangential and normal stress on the layer. Thus, general energy dissipation rate in the transition layer can be expressed as [19]

\[ D = \tau \frac{\partial \eta}{\partial t} - \sigma \frac{\partial \varepsilon}{\partial t} \quad \text{(7)} \]

According to the geometric relationships, one can get

\[ \begin{cases} \frac{\partial \eta}{\partial t} = \frac{\Delta v_t}{\delta} \\ \frac{\partial \varepsilon}{\partial t} = \frac{\Delta v_n}{\delta} \end{cases} \quad \text{(8)} \]

Further, based on the conception of the average stresses on the layer, there is

\[ \begin{cases} \tau = \frac{X_i}{d_i} \\ \sigma = \frac{E_i}{d_i} \end{cases} \quad \text{(9)} \]

Substituting Eqs. (8) and (9) into Eq. (7), one can obtain

\[ D = \frac{X_i \Delta v_t - E_i \Delta v_n}{d_i \delta} \quad \text{(10)} \]

Then, taking into account that the energy dissipation rate in the transition layer should not be negative (\( D \geq 0 \)), one can further get

\[ \frac{X_i}{E_i} \geq \frac{\Delta v_n}{\Delta v_t} \quad \text{(11)} \]

Since the transition layer is not necessarily in the limit state, the shear stress on it is not beyond the related shear strength. Further, the corresponding actual shear strain is not more than the ultimate shear strain. So, one can get

\[ \Delta v_s \leq (\Delta v_t)_{\text{limit}} \quad \text{(12)} \]

According to the admissible failure mechanism of kinematical system of soil mass which can be reasonably used in slope stability analysis [19], the ratio of normal velocity over tangential velocity in the transition layer is the tangent value of dilation angle \( \psi \) of the soil. Thus, Substituting Eq. (12) into Eq. (11) one can obtain

\[ \frac{X_i}{E_i} \geq \frac{\Delta v_n}{\Delta v_t} = \tan \psi \quad \text{(13)} \]

Therefore, Eq. (13) exhibits that the dip angle of the resultant interslice force should be not less than the dilation angle of the soil.
Then, the problem of slope stability analysis is equivalent to solving the minimum value of the factor of safety under the control conditions of Eqs. (1) to (4) and restraint conditions of Eqs. (6) and (13). A potential slide mass is divided into n slices, and for the last slice, there are \( E_{n+1} = 0, X_{n+1} = 0 \) and \( h_{n+1} = 0 \). So for each slice there are five unknowns: \( N_i, T_i, E_i, X_i, \) and \( h_i \). In addition, the factor of safety of slope stability \( F_i \) is another unknown. Consequently, there are totally \( 5n+1 \) unknowns. Based on Eqs. (1) to (4), one can obtain \( 4n \) independent equations. And for the first slice, there are \( E_1 = 0, X_1 = 0 \) and \( h_1 = 0 \). Thus, solving the factor of safety of a slope with potential slide mass divided into \( n \) (\( \geq 3 \)) slices can be actually regarded as an \( n-2 \) dimensional non-linear programming problem to find the minimum factor of safety. As long as the factor of safety is sought out, the other variables including the tangential and normal force on the sides of all slices can be also obtained simultaneously. As for the specifically operating procedure, Sequential Quadratic Programming (SQP) method in MATLAB software can be used to solve this programming problem [20]. There is a locally optimal solution to the problem with constrained variables, and it can be carried out via the inserted fmincon function (Constrained nonlinear minimization) of the optimization tool in MATLAB [20].

3. VERIFICATION EXAMPLES AND DISCUSSION

In order to verify the proposed method, five examples are taken next. Since experiments show that the dilation angle of soil is smaller than the internal friction angle [21], the dilation angle is adopted as \( \psi \), \( 3\psi/4 \), \( \psi/2 \), \( \psi/4 \), and 0, respectively to sufficiently reflect the influence of dilation angle on the analysis results of slope stability.

3.1 Example 1: a homogeneous clay slope with circular slip surface and \( F_s > 1 \)

Fig. 3 shows an example of a soil slope with a circular slip surface and 10 m height [10]. Point \( O_c \) is the rotation center of a potential slip surface, and the potential slide mass of the slope is divided into 9 slices. Point O is the top point of the slip surface. The factors of safety figured out by SQP are exhibited in Table 1, where the symbols of MPM-I, MPM-II, MPM-III, and MPM-IV represent correspondingly four cases that the function \( f(x) \) of interslice force is assumed as constant, custom unimodal curve, custom unimodal curve with two valleys, and straight line, respectively. Besides the results by the non-circular analysis method, circular analysis result (Slip circle analysis) is also listed after Morgenstern and Price (1965) [15]. The results show the maximum relative error between the proposed method and MPM is 5 %. Moreover, the factors of safety obtained by the proposed method are not more than those by the non-circular MPM and circular analysis method. The reason is that the proposed method is actually able to obtain the optimal solution of the factor of safety using the optimization search procedure with considering all constraint equations of slices, rather than introducing the assumption of interslice force function as does the MPM. Therefore, the proposed method is more general and rigorous than existing methods and can theoretically find the minimum value of the factor of safety.

Also, Table 1 shows the factor of safety obtained using the proposed method increases very slightly with the increase of the dilation angle. If the dilation angle varies from 0 to \( \psi \), the factor of safety increases only about 0.2 %. So it means the dilation angle has little effect on the analysis results of slope stability. In fact, a numerical simulation model of the example via FLAC3D (see Fig. 4) is established to reflect further the influence of the dilation angle on the factor of safety. The numerical model consists of 11632 quadrangle elements and 23780 nodes. Perfectly elastoplastic constitutive model and Mohr-Coulomb strength principle are assumed to simulate the slope soil. As shown in Fig. 5, the factor of safety by FLAC3D is marginally increasing with the dilation angle and has only 0.8 % growth if \( \psi = \phi \). The tendency of \( F_s \) varied with \( \psi/\phi \) by the proposed method is in good agreement with that by FLAC3D, which also indicates the proposed method is acceptable.

In order to indicate the directions of interslice force or interslice thrust line obtained by various methods, \( \lambda f(x) \) [10] is still cited here to express the ratio of tangential over normal force on the side of slice.

\[
\lambda f(x) = \frac{X_i}{E_i} \quad (14)
\]
As shown in Fig. 6, there are remarkable differences of the dip angles of interslice forces among the various methods including classical Lowe and Karafiath method (LKM) [6], Corps of Engineers #1 method (CEM1) [7], Corps of Engineers #2 method (CEM2) [7], ITFM [8], and MPM-I [10]. However, the results obtained by the ITFM are relatively close to those calculated using the proposed method. Fig. 7 indicates further that there are distinct alterations of the interslice thrust lines between

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**Table 1. Analysis results of example 1.**

<table>
<thead>
<tr>
<th>Methods</th>
<th>$F_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method ($\psi = \phi$)</td>
<td>2.044</td>
</tr>
<tr>
<td>Proposed method ($\psi = 0.75\phi$)</td>
<td>2.042</td>
</tr>
<tr>
<td>Proposed method ($\psi = 0.5\phi$)</td>
<td>2.041</td>
</tr>
<tr>
<td>Proposed method ($\psi = 0.25\phi$)</td>
<td>2.041</td>
</tr>
<tr>
<td>Proposed method ($\psi = 0$)</td>
<td>2.04</td>
</tr>
<tr>
<td>Slip circle analysis</td>
<td>2.098</td>
</tr>
<tr>
<td>MPM-I</td>
<td>2.045</td>
</tr>
<tr>
<td>MPM-II</td>
<td>2.136</td>
</tr>
<tr>
<td>MPM-III</td>
<td>2.134</td>
</tr>
<tr>
<td>MPM-IV</td>
<td>2.044</td>
</tr>
</tbody>
</table>

Unit weight 20kN/m$^3$
Cohesion 20kPa
Internal friction angle 20°
Elastic modulus 30MPa
Poisson’s ratio 0.4

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**Figure 3. Sketch map of the slope example 1.**

**Figure 4. Numerical simulation model of example 1 via FLAC3D.**

**Figure 5. Relationship between factor of safety and soil dilation angle in example 1.**

**Figure 6. Variation of dip angle of interslice forces in example 1.**

**Figure 7. Variation of ratio of tangential force over normal force on the sides of slices in example 1.**
the proposed method and the MPM. It can be comprehensively seen from Table 1 and Fig. 7 that the more approximate the interslice force function selected in the MPM is to that by the proposed method, the closer the factor of safety by the MPM is to the minimum value or that obtained by the proposed method. In this regard, the proposed method is fairly helpful to conducting the selection of the interslice force function in the MPM.

3.2 Example 2: a homogeneous clay slope with circular slip surface and $F_s$ close to 1

Fig. 8 shows a soil slope with a circular slip surface and 10m height [22], which is one of the examinations of slope stability by Australia Computer Aided Design Society (ACADS). A potential slide mass of the slope is divided into 10 slices. The results of the ratio of tangential force over normal force on the sides of slices are shown in Fig. 9, where MPM-V represents the case that the function of interslice force is semi-sinusoidal curve. It can be seen that the ratios obtained by the proposed method under various dilation angles are almost consistent in trend. For the slices with the higher ratio, the dilation angle has no effect on the ratio; but for the slices with the lower ratio, the dilation angle has only small effect on the ratio. The factor of safety by the proposed method increases only about 0.05% with the increase of dilation angle, and they are very close to the results by the MPM. But the factor of safety under $\psi = 0$ is not more than that by the MPM. Fig. 10 shows the variations of dip angle of interslice forces by several classical slice methods such as the LKM, CEM1, CEM2, ITFM, and MPM-I. The results also indicate that the dip angles assumed by the ITFM are rather closer to those computed by the proposed method.

3.3 Example 3: a nonhomogeneous soil slope with circular slip surface

Fig. 11 shows an example of a nonhomogeneous soil slope with two layers [23]. The potential slide mass of the slope is divided into 9 slices. As shown in Fig. 12, the ratios of tangential force over normal force on the sides of slices obtained by the proposed method are clearly different from those computed by the MPM. Similar to the analysis results of the examples mentioned above, the factor of safety obtained by the proposed method under $\psi = 0$ is the minimum. And it is about 0.1% less than that by the MPM. Also, there are obvious differences of
the dip angles of interslice forces between the proposed method and the classical slice methods, but the dip angles assumed by the ITFM are relatively closer to those obtained using the proposed method (see Fig. 13).

![Figure 12](image1.png)

**Figure 12.** Variation of ratio of tangential force over normal force on the sides of slices in example 3.

![Figure 13](image2.png)

**Figure 13.** Variation of dip angle of interslice forces in example 3.

### 3.4 Example 4: a nonhomogeneous cohesionless slope with non-circular slip surface

As shown in Fig. 14, potential slide mass of a nonhomogeneous cohesionless slope with non-circular slip surface is divided into 5 slices. It can be seen from Fig. 15 the values of \( \lambda_f(x) \) obtained by the proposed method are close to those by MPM-I and MPM-III, but apparently different from those by MPM-II. The proposed factor of safety is almost identical with that by MPM. Moreover, there are observable differences of the dip angles of interslice forces between the proposed method and ITFM as well as CEM2, but the proposed dip angles are fairly closer to those by CEM1 and LKM (see Fig. 16). In particular, the dip angles using the proposed method are between 16° and 20°. Therefore, the calculation results show further the dip angles are between \( \psi \) and \( \varphi \) as demonstrated theoretically above.

![Figure 14](image3.png)

**Figure 14.** Sketch map of the slope example 4.

![Figure 15](image4.png)

**Figure 15.** Variation of ratio of tangential force over normal force on the sides of slices in example 4.

![Figure 16](image5.png)

**Figure 16.** Variation of dip angle of interslice forces in example 4.
3.5 Example 5: a nonhomogeneous clay slope with non-circular slip surface

Fig. 17 shows a nonhomogeneous clay slope with non-circular slip surface and potential slide mass divided into 4 slices. As shown in Fig. 18, \( \lambda f(x) \) by the proposed method are evidently different from those computed by MPM-I, but relatively closer to those by MPM-II and MPM-III. The proposed values are observably influenced by \( \psi \) in the example. Similarly, the factor of safety obtained by the proposed method is very close to that by MPM. Besides, there are still noticeable differences of the dip angles of interslice forces between the proposed method and the classical slice methods (see Fig. 19).

4 CONCLUDING SUMMARY

The general slice method with satisfying all equilibrium conditions of slices for slope stability analysis can be improved further without assuming the direction or magnitude of interslice forces. Based on the admissible failure mechanism of kinematical system of soil mass which can be used in slope stability analysis, it is demonstrated that the dip angles of interslice forces should not be less than the dilation angle of the slope soil. Therefore, the slope stability analysis can be transformed equivalently into a non-linear programming problem to be solved with clear ranges of the directions of interslice forces. Moreover, the closed-form solution of the slice method for slope stability analysis is obtained.

The proposed method is applicable to both homogeneous and nonhomogeneous soil slopes without any limitation of potential slip surfaces. Compared to classical slice methods of limit equilibrium, the proposed method can more rigorously find the minimum value of the factor of safety of slope stability. The effect of the dilation angle of slope soil on the factor of safety is involved in the proposed method, but analysis results of some examples show the dilation angle has little effect on the slope stability. In addition, among the classical slice methods with different assumption of the dip angle of interslice force for stability analysis of slopes with circular slip surfaces, the directions of interslice forces obtained by the ITFM are fairly closer to those computed using the proposed method. However, for slopes with non-circular slip surfaces, there are obvious differences of the interslice force directions between the proposed method and ITFM.

Acknowledgments

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<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>Cohesion of the slope soil</td>
</tr>
<tr>
<td>$d_i$</td>
<td>Height of the interface between slice $i$ and slice $i-1$</td>
</tr>
<tr>
<td>$D$</td>
<td>Internal energy dissipation rate of the thin transition layer between adjacent slices</td>
</tr>
<tr>
<td>$E_i$</td>
<td>Normal force on the interface between slice $i$ and slice $i-1$</td>
</tr>
<tr>
<td>$F_s$</td>
<td>Factor of safety of slope stability</td>
</tr>
<tr>
<td>$h_i$</td>
<td>Height of action point of the interslice force between slice $i$ and slice $i-1$</td>
</tr>
<tr>
<td>$i$</td>
<td>Number of the $i$th slice</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Development coefficient of shear limit resistance on the interface between slice $i$ and slice $i-1$</td>
</tr>
<tr>
<td>$l_i$</td>
<td>Length of the bottom of slice $i$</td>
</tr>
<tr>
<td>$n$</td>
<td>Total number of slices of the potential slide mass of a soil slope vertically divided</td>
</tr>
<tr>
<td>$N_i$</td>
<td>Normal force on the bottom of slice $i$</td>
</tr>
<tr>
<td>$t$</td>
<td>Minimal time</td>
</tr>
<tr>
<td>$T_i$</td>
<td>Tangential force on the bottom of slice $i$</td>
</tr>
<tr>
<td>$W_i$</td>
<td>Weight of slice $i$</td>
</tr>
<tr>
<td>$x$</td>
<td>Horizontal coordinate with respect to the origin O</td>
</tr>
<tr>
<td>$X_i$</td>
<td>Tangential force on the interface between slice $i$ and slice $i-1$</td>
</tr>
<tr>
<td>$y$</td>
<td>Vertical coordinate with respect to the origin O</td>
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<tr>
<td>$\alpha_i$</td>
<td>Dip angle of the bottom of slice $i$</td>
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<tr>
<td>$\gamma$</td>
<td>Unit weight of slope soil</td>
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<tr>
<td>$\delta$</td>
<td>Thickness of the thin transition layer between adjacent slices</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Normal strain in the thin transition layer</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Internal friction angle of the slope soil</td>
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<tr>
<td>$\eta$</td>
<td>Shear strain in the thin transition layer</td>
</tr>
<tr>
<td>$\lambda f(x)$</td>
<td>Assumed function of the ratio of tangential force $X_i$ over normal force $E_i$ of slices, where $\lambda$ is a dimensionless coefficient</td>
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<tr>
<td>$\sigma$</td>
<td>Average normal stress on the thin transition layer</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Average shear stress on the thin transition layer</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>Shear stress on the thin transition layer in the plastic limit state</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Dilation angle of the slope soil</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Resultant velocity of a rigid slice, where subscript $i$ and $i-1$ denote slice $i$ and slice $i-1$, respectively</td>
</tr>
<tr>
<td>$\Delta v_s$</td>
<td>Tangential velocity of slice $i$ relative to slice $i-1$</td>
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<tr>
<td>$\Delta v_n$</td>
<td>Normal velocity of slice $i$ relative to slice $i-1$</td>
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<tr>
<td>$(\Delta v)_\text{limit}$</td>
<td>Tangential velocity of slice $i$ relative to slice $i-1$ in the plastic limit state</td>
</tr>
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### REFERENCES


